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Radiometric Calibration of Gray Scale Images

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Abstract

This paper describes a process for the radiometric calibration of gray scale images and linear transformations of gray scale images. We begin by defining radiometric calibration and show that it can be posed as a linear least squares problem. We then consider radiometric calibration using linear transformations of images. When the transform coefficients are real valued we show that the calibration problem is still a linear least squares problem. When the transform coefficients are complex valued the calibration problem is a rather simple non-linear least squares problem. We briefly discuss existing optimization software for solving both linear and non-linear least squares problems.

1 Radiometric Calibration

It is well known that there is often a difference between the actual intensity of incident radiation and the intensity measured by a detector. This difference is due to nonlinearity of the detector response, and noise associated with the detection mechanism. This means that the relationship between measured and actual intensity is often non-linear. In the absence of proper detector characterization, the relationship is also unknown.

In this paper we consider a portion of the process involved in comparing two images. Often, we want image comparison to be invariant to intensity differences between the two images. For example if we have two images of the same object, but all the intensities in one image are twice those in the other image, we often want to say that these images are identical. Unfortunately, the relationship between the intensities of images is often non-linear and usually unknown. We call the process of calibrating two images the *radiometric calibration problem*.

2 Problem Formulation

In this work we define an *image* to be a set $I = \{(v,(i,j)) : v \in \{0,1,\ldots,255\}, \quad i = 1,2,\ldots,P, \quad j = 1,2,\ldots,Q\}$. The pair $(v,(i,j))$ is a *pixel* where v is the value and the pair (i,j) is the location. As a shorthand notation, we denote $v_I(i,j)$ as the value v of the pixel at location (i,j) in image I. All images considered in this work are gray scale, as seen from the definition $v \in \{0,1,\ldots,255\}$. We assume we are given two images of the same size $P \times Q$, denoted I_s and I_m.

One way to calibrate the images I_s and I_m is as follows. Consider all pixels in image I_m with value $v = k$ and construct the set of locations $L_{I_m}(k)$ defined by $L_I(k) = \{(i,j) : v_I(i,j) = k\}$. Clearly $L_I(k)$ is the set of all pixel locations in image I which have pixel value k. For all pixels in $L_{I_m}(k)$ find a new pixel value ϕ_k such that the sum of the squared difference between ϕ_k and the pixel values at locations $L_{I_m}(k)$ in image I_s is minimized. Repeat this procedure for all pixel values in image I_m. This problem can formally be written as

$$\hat{\boldsymbol{\phi}} = \arg\min_{\boldsymbol{\phi}} \sum_{k=0}^{255} \sum_{(i,j)\in L_{I_m}(k)} \big(v_{I_s}(i,j) - \phi_k\big)^2, \tag{1}$$

where $\boldsymbol{\phi}$ is a vector whose kth element is ϕ_k and $v_{I_s}(i,j)$ is the value of the pixel at location (i,j) in image I_s. This problem can be rewritten in the standard form for a linear least squares problem by introducing the following notation. Define the *image matrix* $\boldsymbol{M}_I$ as a $P \times Q$ matrix of integers in $\{0,1,\ldots,255\}$ such that $[\boldsymbol{M}_I]_{ij} = v_I(i,j)$. Define the *image vector* $\boldsymbol{m}_I$ as a PQ vector of integers in $\{0,1,\ldots,255\}$ formed by row ordering $\boldsymbol{M}_I$. Define $\boldsymbol{B}_I$ as a $PQ \times 256$ matrix of binary values in $\{0,1\}$ such that row $l = (Q-1)i + j$ contains a 1 in column $v_I(i,j)$ and a 0 in all other columns. Using this notation, Equation (1) can be rewritten as

$$\begin{aligned} \hat{\boldsymbol{\phi}} &= \arg\min_{\boldsymbol{\phi}} \left\langle \, (\boldsymbol{m}_{I_s} - \boldsymbol{B}_{I_m}\boldsymbol{\phi}), (\boldsymbol{m}_{I_s} - \boldsymbol{B}_{I_m}\boldsymbol{\phi}) \, \right\rangle, \\ &= \arg\min_{\boldsymbol{\phi}} \Big(\boldsymbol{\phi}^\top \boldsymbol{B}_{I_m}^\top \boldsymbol{B}_{I_m} \boldsymbol{\phi} - 2\, \boldsymbol{m}_{I_s}^\top \boldsymbol{B}_{I_m} \boldsymbol{\phi} + \boldsymbol{m}_{I_s}^\top \boldsymbol{m}_{I_s} \Big), \end{aligned} \tag{2}$$

where $\langle \boldsymbol{v}, \boldsymbol{v} \rangle$ is the *inner product* of two vectors. As an aside, recall that $\boldsymbol{A}^\mathsf{T}\boldsymbol{A}$ has the properties: it is real valued, square, symmetric, positive semi-definite, and has the same nullspace as $\boldsymbol{A}$. Recall that the *rank* of $\boldsymbol{A}$ equals the number of linearly independent rows which equals the number of linearly independent columns. Also the dimension of the null space of $\boldsymbol{A}$ is the number of columns minus the rank. Furthermore if $\boldsymbol{A}$ is full rank then $(\boldsymbol{A}^\mathsf{T}\boldsymbol{A})^{-1}$ exists and the linear least squares problem has a *unique* solution. Clearly the rank r of $\boldsymbol{B}_{I_m}$ is equal to the number of different pixel values appearing in image I_m, so $r \le 256$. The matrix $\boldsymbol{B}_{I_m}$ can always be made full rank by removing all columns $\boldsymbol{b}_k$ such that $\boldsymbol{b}_k = \boldsymbol{0}$. We must also remove the corresponding elements ϕ_k from $\boldsymbol{\phi}$. This means that Equation (2) always has a unique solution.

Suppose that we have a real valued linear transformation of the form

$$t_I(k,l) = \sum_{i=1}^{P} \sum_{j=1}^{Q} c_{k,l}(i,j)\, v_I(i,j), \quad k = 1, \ldots, R \le P, \quad l = 1, \ldots S \le Q, \tag{3}$$

where $c_{k,l}(i,j)$ are the real valued coefficients of the transform. The transform calibration problem is to find a set of pixels values $\boldsymbol{\phi}$ such that the squared difference between the transformed images is minimized, in other words

$$\hat{\boldsymbol{\phi}} = \arg\min_{\boldsymbol{\phi}} \sum_{k=1}^{R} \sum_{l=1}^{S} \big(t_{I_s}(k,l) - t_{\boldsymbol{B}_{I_m}\boldsymbol{\phi}}(k,l) \big)^2. \tag{4}$$

Define $\boldsymbol{C}_{k,l}$ as a $P \times Q$ matrix of transform coefficients for fixed k and l and $\boldsymbol{c}_d(k,l)$ as a $P\,Q$ vector formed by row ordering $\boldsymbol{C}_{k,l}$. Define $\boldsymbol{T}$ as an $R\,S \times P\,Q$ matrix whose dth row is the vector $\boldsymbol{c}_d(k,l)^\mathsf{T}$. Using this notation Equation (4) can be rewritten as

$$\begin{aligned} \hat{\boldsymbol{\phi}} &= \arg\min_{\boldsymbol{\phi}} \big\langle \boldsymbol{T}\big(\boldsymbol{m}_{I_s} - \boldsymbol{B}_{I_m}\boldsymbol{\phi}\big), \boldsymbol{T}\big(\boldsymbol{m}_{I_s} - \boldsymbol{B}_{I_m}\boldsymbol{\phi}\big) \big\rangle, \\ &= \arg\min_{\boldsymbol{\phi}} \Big(\boldsymbol{\phi}^\mathsf{T}\boldsymbol{B}_{I_m}^\mathsf{T}\boldsymbol{T}^\mathsf{T}\boldsymbol{T}\boldsymbol{B}_{I_m}\boldsymbol{\phi} - 2\,\boldsymbol{m}_{I_s}^\mathsf{T}\boldsymbol{T}^\mathsf{T}\boldsymbol{T}\boldsymbol{B}_{I_m}\boldsymbol{\phi} + \boldsymbol{m}_{I_s}^\mathsf{T}\boldsymbol{T}^\mathsf{T}\boldsymbol{T}\boldsymbol{m}_{I_s} \Big). \end{aligned} \tag{5}$$

The problem in Equation (5) is also a standard linear least squares problem. Note that we can use two different transformations $\boldsymbol{T}_{I_s}$ and $\boldsymbol{T}_{I_m}$ for the two images and still have the linear least squares problem

$$\hat{\boldsymbol{\phi}} = \arg\min_{\boldsymbol{\phi}} \big\langle \big(\boldsymbol{T}_{I_s}\boldsymbol{m}_{I_s} - \boldsymbol{T}_{I_m}\boldsymbol{B}_{I_m}\boldsymbol{\phi}\big), \big(\boldsymbol{T}_{I_s}\boldsymbol{m}_{I_s} - \boldsymbol{T}_{I_m}\boldsymbol{B}_{I_m}\boldsymbol{\phi}\big) \big\rangle. \tag{6}$$

Consider the properties of the solution for Equations (5) and (6). Let $\boldsymbol{A} = \boldsymbol{T}\,\boldsymbol{B}_{I_m}$, which is an $R\,S \times 256$ matrix. If the columns of $\boldsymbol{A}$ are linearly independent, then Equation (5) (or (6)) has a unique solution. Note that the columns of $\boldsymbol{A}$ can *not* be linearly independent if $R\,S < 256$. If the columns of $\boldsymbol{A}$ are *not* linearly independent, then there are an infinite number of solutions to Equation (5) (or (6)). Specifically, given a particular solution $\hat{\boldsymbol{\phi}}$ to Equation (5) (or (6)), the sum of $\hat{\boldsymbol{\phi}}$ and any element in the nullspace of $\boldsymbol{A}$ is also a solution. Recall that the nullspace of $\boldsymbol{A}$ consists of all $\boldsymbol{\phi}$ such that $\boldsymbol{A}\,\boldsymbol{\phi} = \boldsymbol{0}$.

Suppose that we have a complex valued linear transformation of the form

$$\bar{t}_I(k,l) = \sum_{i=1}^{P} \sum_{j=1}^{Q} \bar{c}_{k,l}(i,j)\, v_I(i,j), \quad k = 1, \ldots, R \le P, \quad l = 1, \ldots S \le Q, \tag{7}$$

where $\bar{c}_{k,l}(i,j)$ are the complex valued coefficients of the transform. The transform calibration problem is to find a set of pixels values $\boldsymbol{\phi}$ such that the squared difference between the *magnitudes* of transformed images is minimized. This problem can be written

$$\hat{\boldsymbol{\phi}} = \arg\min_{\boldsymbol{\phi}} \sum_{k=1}^{R} \sum_{l=1}^{S} \Big(\big|\bar{t}_{I_s}(k,l)\big| - \big|\bar{t}_{\boldsymbol{B}_{I_m}\boldsymbol{\phi}}(k,l)\big| \Big)^2. \tag{8}$$

where $\big|\bar{t}_I(k,l)\big| = \sqrt{\bar{t}_I(k,l)\,\bar{t}_I^*(k,l)}$ is the magnitude of $\bar{t}_I(k,l)$ and $\bar{t}_I^*(k,l)$ is the complex conjugate of $\bar{t}_I(k,l)$. Therefore $\big|\bar{t}_I(k,l)\big|$ can be written as

$$\big|\bar{t}_I(k,l)\big| = \sqrt{\sum_{i=1}^{P} \sum_{j=1}^{Q} \bar{c}_{k,l}(i,j)\,\bar{c}_{k,l}^*(i,j)\,v_I^2(i,j)} \tag{9}$$

Making use of our previous notation, Equation (8) can be rewritten as

$$\hat{\boldsymbol{\phi}} = \arg\min_{\boldsymbol{\phi}} \left\langle \left(\sqrt{\bar{\boldsymbol{T}} \circ \bar{\boldsymbol{T}}^* \boldsymbol{m}_{I_s}^2} - \sqrt{\bar{\boldsymbol{T}} \circ \bar{\boldsymbol{T}}^* (\boldsymbol{B}_{I_m}\boldsymbol{\phi})^2}\right), \left(\sqrt{\bar{\boldsymbol{T}} \circ \bar{\boldsymbol{T}}^* \boldsymbol{m}_{I_s}^2} - \sqrt{\bar{\boldsymbol{T}} \circ \bar{\boldsymbol{T}}^* (\boldsymbol{B}_{I_m}\boldsymbol{\phi})^2}\right) \right\rangle, \tag{10}$$

where the *Schur product* is $[\boldsymbol{A} \circ \boldsymbol{B}]_{ij} = [\boldsymbol{A}]_{ij}\,[\boldsymbol{B}]_{ij}$, $\boldsymbol{A}^*$ is the complex conjugate of $\boldsymbol{A}$, the square of a vector is $\boldsymbol{v}^2 = (v_1^2\ v_2^2\ \cdots\ v_k^2)$, and the square root of a vector is $\sqrt{\boldsymbol{v}} = (\sqrt{v_1}\ \sqrt{v_2}\ \cdots\ \sqrt{v_k})$. Note that Equation (10) is a least squares problem, but it is *not* linear in $\boldsymbol{\phi}$. Recall that each row of $\boldsymbol{B}_{I_m}$ contains a single 1 and the remaining entries are all 0. Therefore $[\boldsymbol{b}_0\,\phi_0]_i + \cdots + [\boldsymbol{b}_{255}\,\phi_{255}]_i = \phi_k$ for some $k \in \{0,\ldots,255\}$ and for all $i = 1,\ldots,P\,Q$, where $\boldsymbol{b}_l$ is the lth column of $\boldsymbol{B}$. This implies that $(\boldsymbol{B}\boldsymbol{\phi})^2 = \boldsymbol{B}(\boldsymbol{\phi})^2$.

3 Solution Methods

The calibration problems defined by Equations (2), (5), and (6) are linear least squares problems. The algorithm LSQR developed by Paige and Saunders (1982) has been shown in practice to solve linear least squares problems both quickly and accurately. Specifically the algorithm solves the damped least squares problem $\min_{\boldsymbol{x}} \|\boldsymbol{b} - \boldsymbol{A}\,\boldsymbol{x}\|_2^2 + \|\lambda \boldsymbol{x}\|_2^2$ where $\boldsymbol{A}$ has m rows and n columns and $\lambda \geq 0$. LSQR is a variant of the well known conjugate gradient (CG) algorithm based on a bidiagonalization procedure developed by Golob and Kahan. It generates a sequence of approximate solutions $\{\boldsymbol{x}_k\}$ such that the residual norm $\|\boldsymbol{r}_k\|_2 = \sqrt{\|\boldsymbol{b} - \boldsymbol{A}\,\boldsymbol{x}_k\|_2^2 + \|\lambda \boldsymbol{x}_k\|_2^2}$ decreases monotonically. Analytically LSQR generates the same sequence $\{\boldsymbol{x}_k\}$ as CG, but LSQR is numerically more reliable than CG. LSQR incorporates reliable stopping criteria and computes estimates of $\boldsymbol{x}$, $\|\boldsymbol{x}\|_2$, $\boldsymbol{r}$, $\|\boldsymbol{r}\|_2$, $\big\|\boldsymbol{A}^{\mathsf{T}}\boldsymbol{r}\big\|_2$, the Frobenius norm $\|\boldsymbol{A}\|_F$, the condition number of $\boldsymbol{A}$, the projection $\boldsymbol{A}\,\boldsymbol{x}$, and the standard errors $s_i^2 = \big(\|\boldsymbol{r}\|_2^2 \,/\, \max((m-n),1)\big)[(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A})^{-1}]_{ii}$ where $i = 1,\ldots,n$.

The calibration problem defined by Equation (10) is a nonlinear least squares problem. The standard form for nonlinear least squares problems is

$$\min_{\boldsymbol{x}} \langle\, \boldsymbol{r}(\boldsymbol{x}), \boldsymbol{r}(\boldsymbol{x}) \,\rangle\,. \tag{11}$$

Equation (10) can be expressed in this form by defining $\boldsymbol{b} = \sqrt{\bar{\boldsymbol{T}} \circ \bar{\boldsymbol{T}}^* \boldsymbol{m}_{I_s}^2}$, $\boldsymbol{A} = (\bar{\boldsymbol{T}} \circ \bar{\boldsymbol{T}}^*)\boldsymbol{B}_{I_m}$ and $\boldsymbol{x} = \boldsymbol{\phi}$. Using this notation, the residual is $\boldsymbol{r} = \boldsymbol{b} - \sqrt{\boldsymbol{A}\boldsymbol{x}^2}$ and the Jacobian is $\boldsymbol{J} = (\nabla r_1 \; \nabla r_2 \; \cdots \; \nabla r_{RS})$. Specifically the elements of the Jacobian are

$$J_{ij} = \frac{\partial r_j}{\partial x_i} = \frac{-a_{ji}\, x_i}{\sqrt{a_{j1}\, x_1^2 + a_{j2}\, x_2^2 + \cdots + a_{j256}\, x_{256}^2}} \quad i = 1, \ldots, 256, \quad j = 1, \ldots RS. \tag{12}$$

The Levenberg-Marquardt (LM) algorithm (Levenberg (1944) and Marquardt (1963)) is one traditional method for solving non-linear least squares problems. LM is a variant of the well known Gauss-Newton algorithm in which Equation (11) is approximated by a series of quadratic problems

$$\begin{aligned} &\min_{\boldsymbol{\delta}_k} \;\; \boldsymbol{\delta}_k^\intercal \boldsymbol{J}_k^\intercal \boldsymbol{J}_k \boldsymbol{\delta}_k + 2\, \boldsymbol{r}_k^\intercal \boldsymbol{J}_k^\intercal \boldsymbol{\delta}_k \\ &\text{subject to} \\ &\boldsymbol{\delta}_k^\intercal \boldsymbol{\delta}_k \le \eta_k^2 \;\; , \end{aligned} \tag{13}$$

where $\boldsymbol{\delta}_k$ is found by solving the regularized linear system

$$\big(\boldsymbol{J}_k^\intercal \boldsymbol{J}_k + \lambda \boldsymbol{I}\big)\boldsymbol{\delta}_k = -\boldsymbol{J}_k \boldsymbol{r}_k \;\; , \tag{14}$$

for $\lambda \ge 0$ and $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{\delta}_k$. An excellent implementation of LM due to Moré (1978) appears in MINPACK. Another algorithm that has achieved good results in practice on non-linear least squares problems is NL2SOL by Dennis, Gay, and Welsch (1981).

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